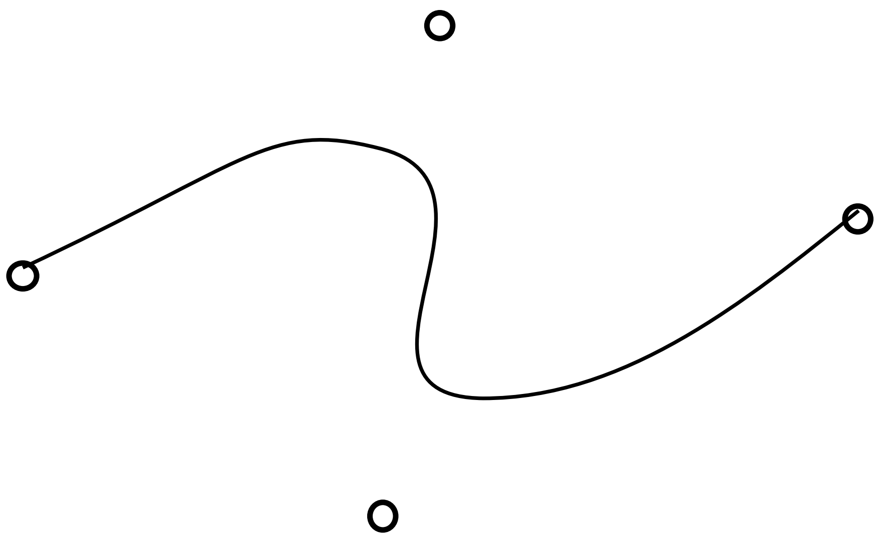
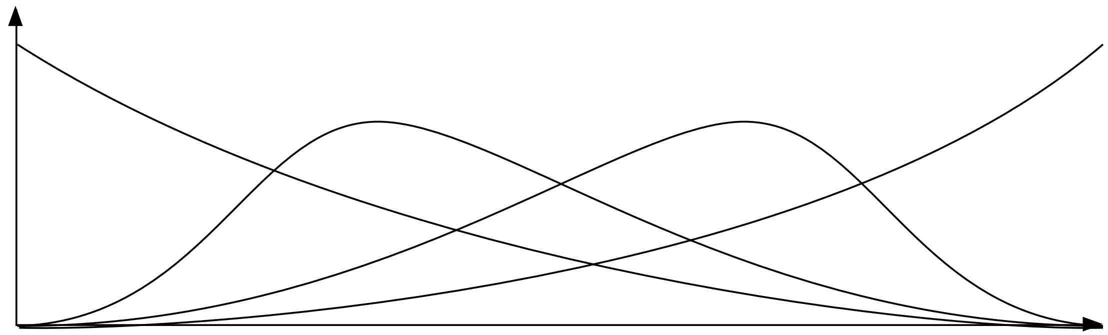
Bézier curves

Typically uses 4 control points per section



Bézier curves

The 4 points are blended together using 4 blending functions

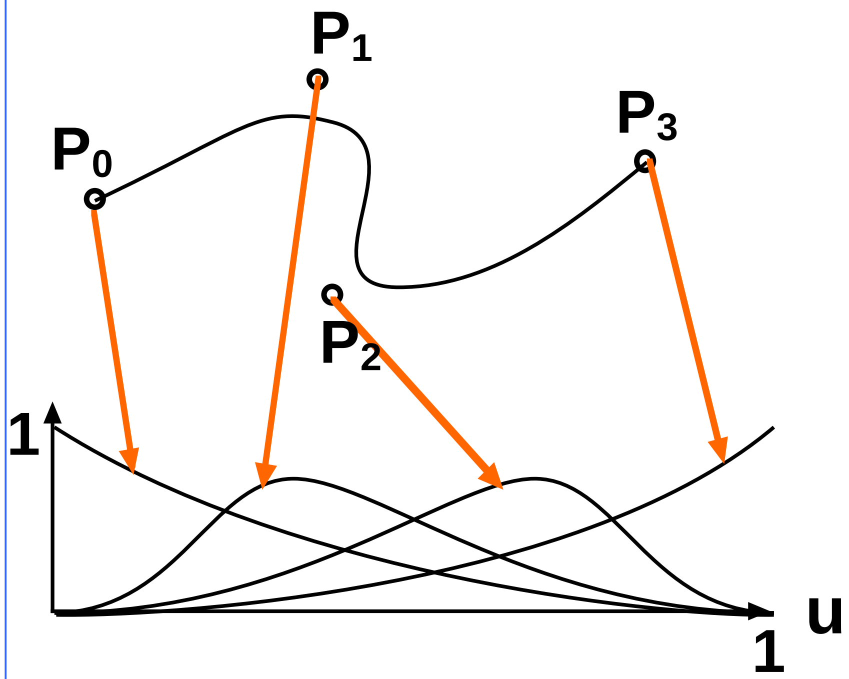


Bézier curves

Blending functions:
Bernstein polynomials

$$BEZ_{0,3} = (1-u)^3$$
$$BEZ_{1,3} = 3u(1-u)^2$$
$$BEZ_{2,3} = 3(1-u)u^2$$
$$BEZ_{3,3} = u^3$$

The sum is 1 for any u



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$$BEZ_{1,3} = 3u(1-u)^2u$$
$$BEZ_{2,3} = 3(1-u)u^2$$
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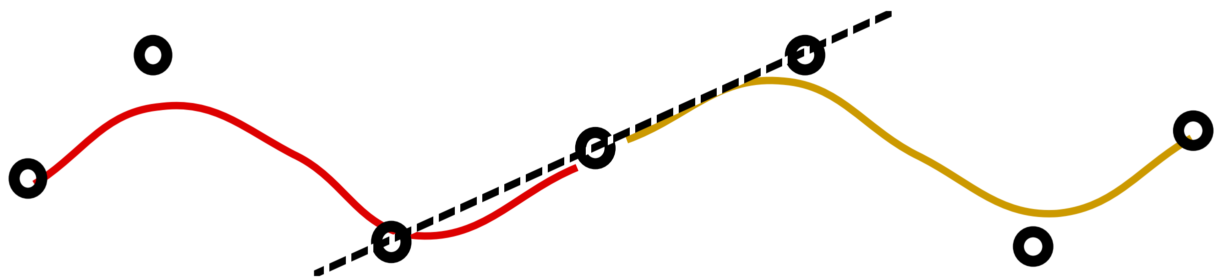
$$P(u) = P_0*(1-u)^3 + P_1*3u(1-u)^2 + P_2*3(1-u)u^2 + P_3*u^3$$
$$= \sum_{i=0}^{3} P_i * BEZ_{i,3}(u)$$

Fitting together sections

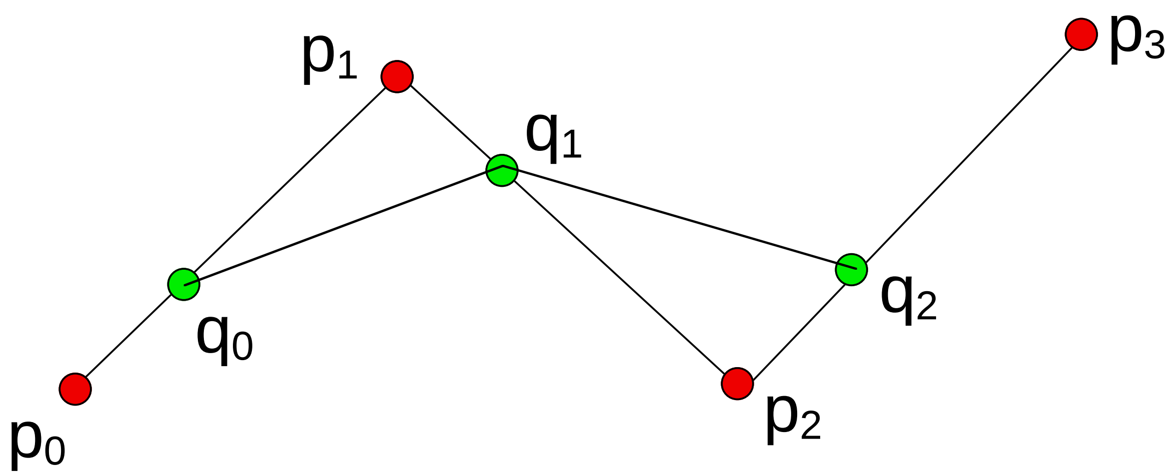
G_0 continuity: just fit the points

G_1 continuity: Make sure the tangents are equal along the edge.
Simple method: Put 3 points in a line



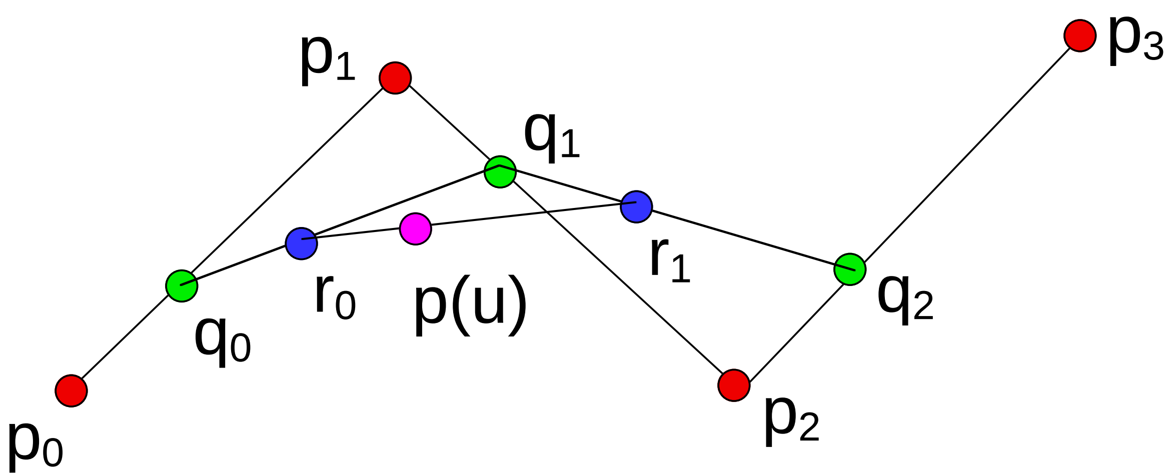
de Casteljau's algorithm

A Bézier is really an *interpolation of interpolations*!



de Casteljau's algorithm

Linear interpoations of linear interpolations until only one point remains



de Casteljau’s algorithm

Gives us the Bernstein polynomials of any level we want.

Linear (2 points) = plain interpolation
Cubic Béziers (3 points)
Quadratic (4 points)
Higher levels possible but not practical

de Casteljau’s algorithm

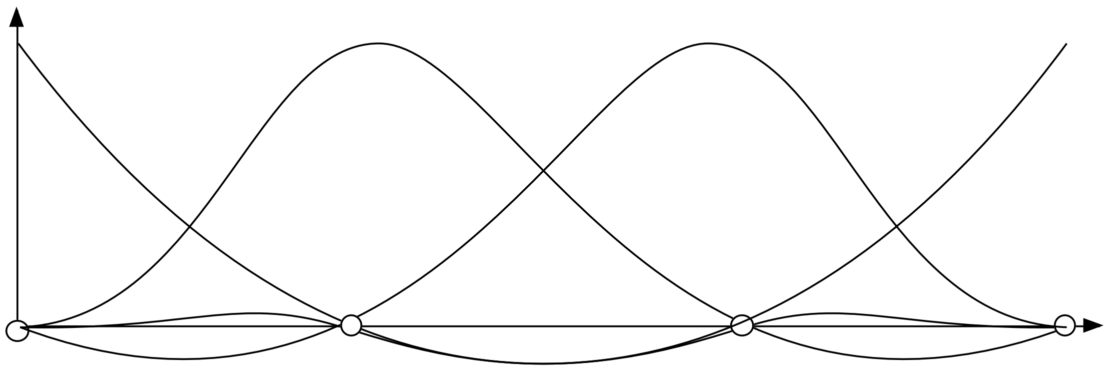
Obvious from figure/method:

- Bézier is always inside convex hull

- Fit together sections by keeping points along a line also obvious - we must start along the tangent!

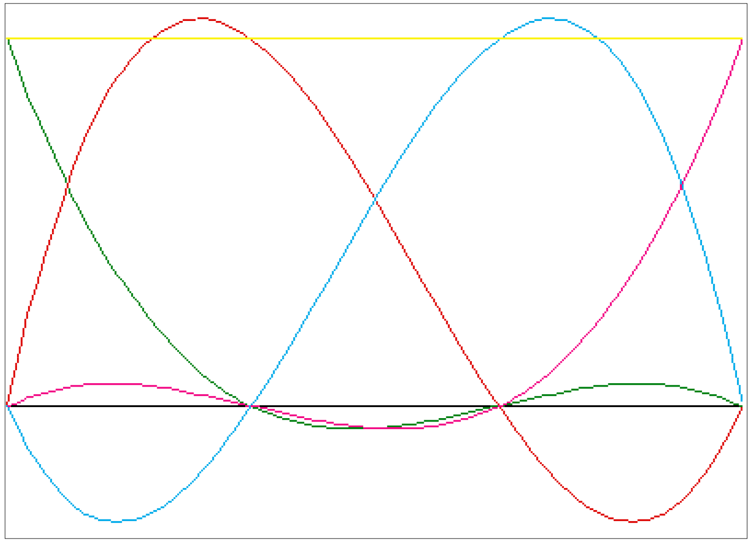
Blending functions for interpolation spline

The points are *blended* together using blending functions



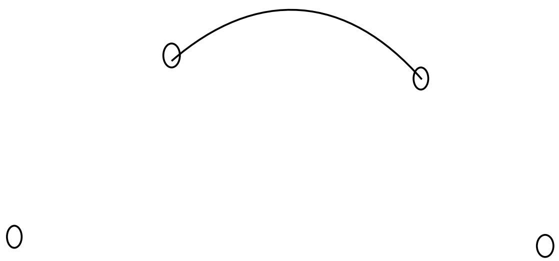
Blending functions for interpolation spline

All blending functions are zero or 1 at the control points!



Actual blending functions for interpolated spline of 4 control points (similar to Bézier)

Cardinal splines
Catmull-Rom splines



Interpolation spline

Specified *only* by control points

Calculated from 4 control points, define between the middle two!

A tension parameter t can adjust the shape

t = 0 => Catmull-Rom

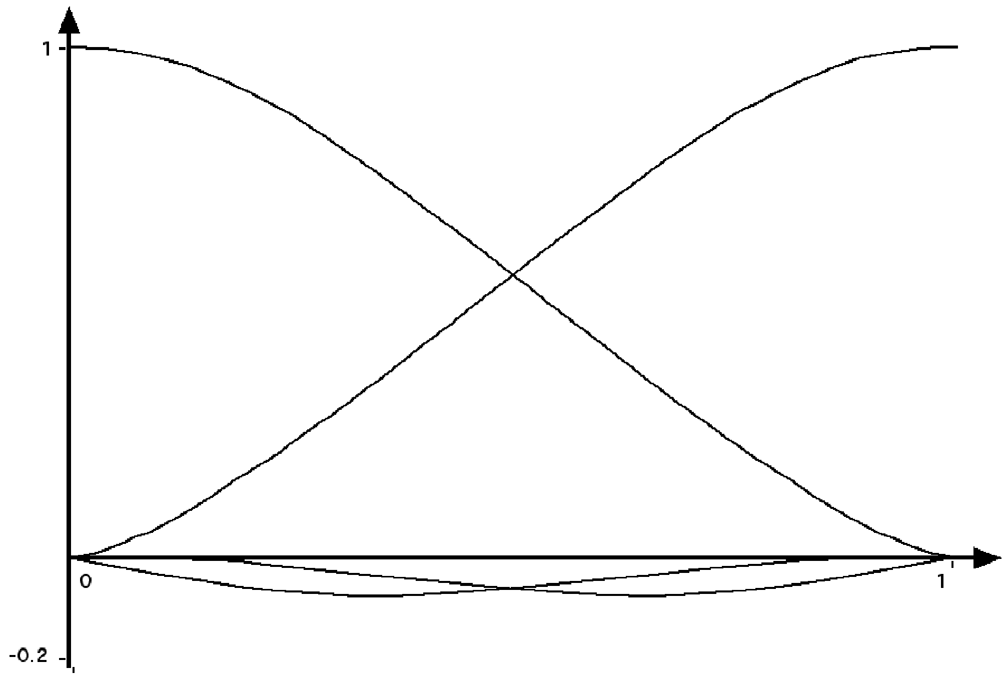
Catmull-Rom splines,
Matrix form

$$P(u) = [\,u^3\ u^2\ u\ 1\,] \begin{bmatrix} -1/2 & 3/2 & -3/2 & 1/2 \\ 1 & -5/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

$$\begin{aligned} P(u) = \quad & p_{k-1}\,(-u^3/2 + u^2 - u/2) + \\ & p_k\,(3u^3/2 - 5u^2/2 + 1) + \\ & p_{k+1}\,(-3u^3/2 + 2u^2 + u/2) + \\ & p_{k+2}\,(u^3/2 - u^2/2) \end{aligned}$$

$$= \quad p_{k-1}{*}CAR_0(u) + p_k{*}CAR_1(u) + p_{k+1}{*}CAR_2(u) + p_{k+2}{*}CAR_3(u)$$

Catmull-Rom splines,
Blending functions



NURBs/NURBS

Non-Uniform Rational B-spline.

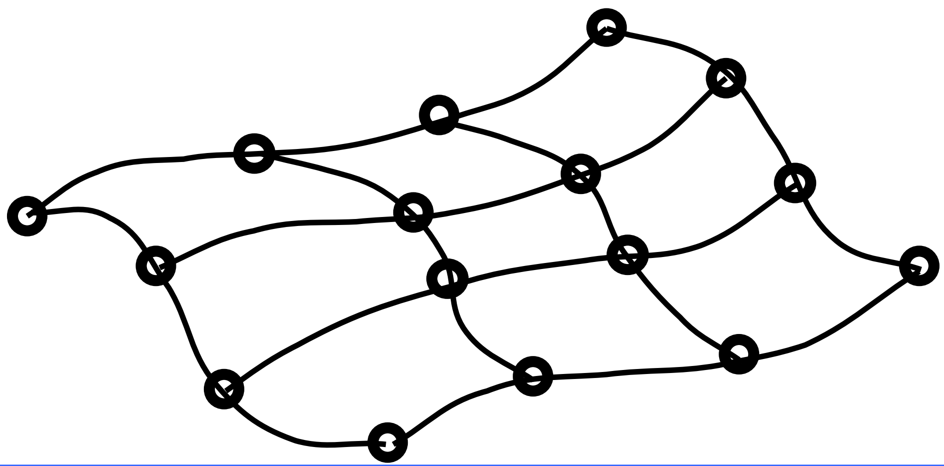
Popular in CAD programs.

Can exactly represent all quadric curves.

Bézier surfaces

A surface is built from a set of Bézier patches

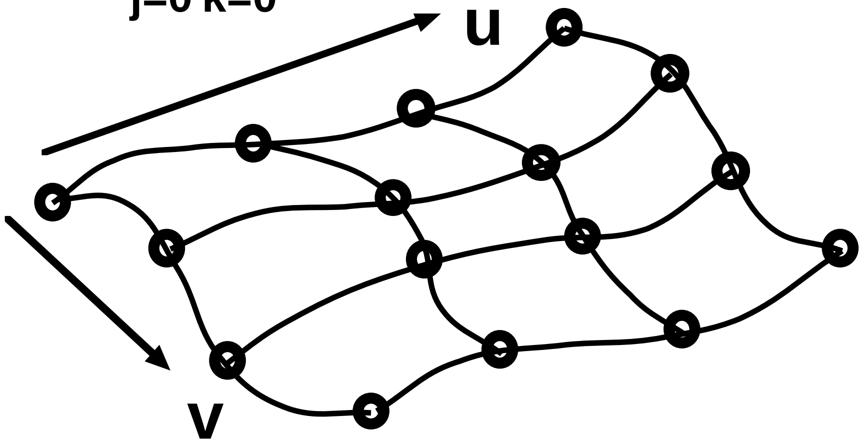
A Bézier patch consists of 16 control points in a 4x4 grid



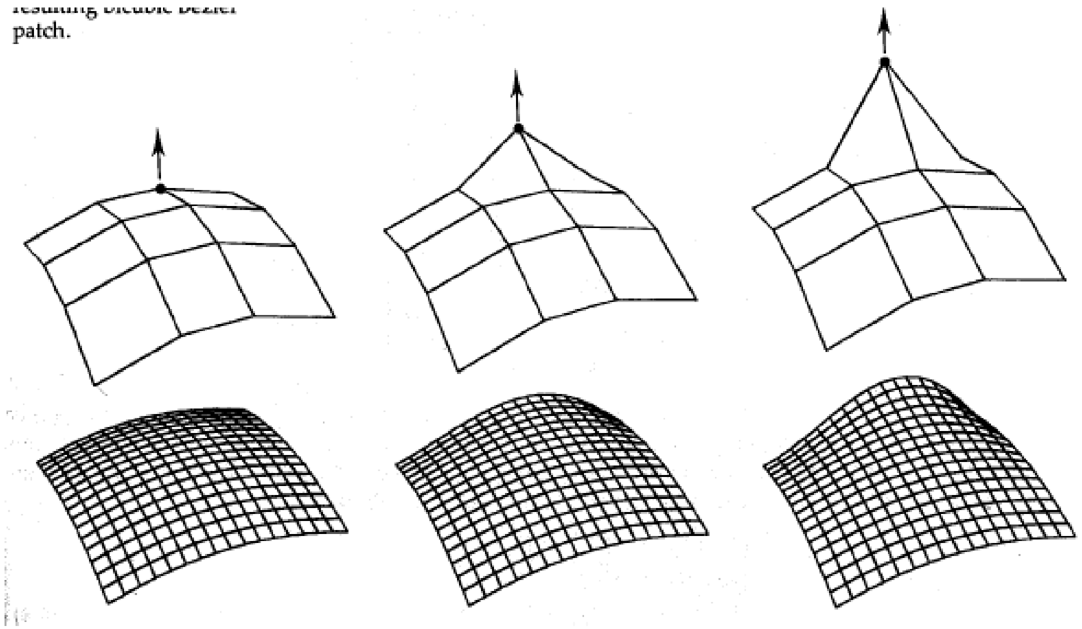
Bézier surfaces

Blending of the 16 control points as a 2-dimensional sum

$$P(u,v) = \sum_{j=0}^{3} \sum_{k=0}^{3} p_{j,k}\, BEZ_{j,3}(v)\, BEZ_{k,3}(u)$$



Bézier surface example



Fitting together patches

Fit in both u and v direction

Make a 3x3 “joystick” at each corner